Lemma 4.1 Assume $C(A_i)$ is the bits needed to encode A_i, if $A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$, then $C(A_1) + C(A_2) \leqslant C(A)$.

$$C(A) = \log[\binom{n(A)}{n_1(A)} \prod_{m=1}^{Nw} \binom{n_1(A) - \sum_{l=1}^{m-1} X_l^A}{X_m^A}]$$

Proof:

$$= \log[\binom{n(A_1)+n(A_2)}{n_1(A_1)+n_1(A_2)} \prod_{m=1}^{Nw} \binom{n_1(A_1) - \sum_{l=1}^{m-1} X_l^{A_1} + n_1(A_2) - \sum_{l=1}^{m-1} X_l^{A_2}}{X_m^{A_1} + X_m^{A_2}}]$$

$$= \log\left[\sum_{x=0}^{n_1(A)} \binom{n(A_1)}{n_1(A)-x}\binom{n(A_2)}{x} \prod_{m=1}^{Nw} \left[\sum_{x=0}^{X_m^A} \binom{n_1(A_1) - \sum_{l=1}^{Nw-1} X_l^{A_1}}{X_m^A - x}\binom{n_1(A_2) - \sum_{l=1}^{Nw-1} X_l^{A_2}}{x}\right]\right]$$

$$\geq \log\left[\binom{n(A_1)}{n_1(A_1)}\binom{n(A_2)}{n_1(A_2)} \prod_{m=1}^{Nw} \left[\binom{n_1(A_1) - \sum_{l=1}^{Nw-1} X_l^{A_1}}{X_m^A - x}\binom{n_1(A_2) - \sum_{l=1}^{Nw-1} X_l^{A_2}}{x}\right]\right]$$

$$= \log\left[\binom{n(A_1)}{n_1(A_1)} \prod_{m=1}^{Nw} \binom{n_1(A_1) - \sum_{l=1}^{m-1} X_l^{A_1}}{X_m^{A_1}}\right] + \log\left[\binom{n(A_2)}{n_1(A_2)} \prod_{m=1}^{Nw} \binom{n_1(A_2) - \sum_{l=1}^{m-1} X_l^{A_2}}{X_m^{A_2}}\right]$$

$$= C(A_1) + C(A_2)$$